On some nonlinear extensions of the angular momentum algebra

C Quesne†‡

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Bd. du Triomphe, B1050 Brussels, Belgium

Short title: *Nonlinear extensions of angular momentum algebra*

PACS number(s): 02.20.+b, 03.65.Fd

† **Directeur de recherches FNRS**
‡ **E-mail: cquesne@ulb.ac.be**

Abstract. Deformations of the Lie algebras so(4), so(3,1), and e(3) that leave their so(3) subalgebra undeformed and preserve their coset structure are considered. It is shown that such deformed algebras are associative for any choice of the deformation parameters. Their Casimir operators are obtained and some of their unitary irreducible representations are constructed. For vanishing deformation, the latter go over into those of the corresponding Lie algebras that contain each of the so(3) unitary irreducible representations at most once. It is also proved that similar deformations of the Lie algebras su(3), sl(3,$\mathbb{R}$), and of the semidirect sum of an abelian algebra t(5) and so(3) do not lead to associative algebras.

1. Introduction

In recent years, many works have been devoted to the study of deformations and extensions of Lie algebras and their applications in various branches of physics. Some of them are carried out in the mathematically well-defined framework of quasitriangular Hopf algebras and deal with the so-called quantum groups and q-algebras (Drinfeld 1986, Jimbo 1985). Others put less emphasis on the coalgebra structure, which is often dropped completely, but instead insist on preserving some other property of the Lie algebra that is deformed. In this second category, one finds for instance some deformed algebras that can be realized in terms of deformed creation and annihilation operators (Fairlie and Zachos 1991, Fairlie and Nuyts 1994).

In the same class, there are also deformed algebras that have a coset structure $g_d = h + v_d$, and can be viewed as nonlinear extensions of an ordinary Lie algebra h (Roček 1991). This means that their generators can be separated into the generators E_i of h and some operators E_α transforming as a representation of h, and commuting among themselves to give a function of the E_i's only. In other words, they satisfy the commutation relations

$$[E_i, E_j] = c_{ij}^k E_k \qquad [E_i, E_\alpha] = (\tau_i)_\alpha^\beta E_\beta \qquad [E_\alpha, E_\beta] = f_{\alpha\beta}(E_i) \tag{1.1}$$

where c_{ij}^k and $(\tau_i)_\alpha^\beta$ are the structure constants and some matrix representation of h respectively, while $f_{\alpha\beta}(E_i)$ are formal power series in the E_i's. The latter are constrained by the associativity requirement, i.e., Jacobi identities, and by the condition that for some limiting values of the parameters, g_d goes over into some Lie algebra g with coset structure $g = h + v$. The simplest example, corresponding to $h = \mathrm{u}(1)$, and $g = \mathrm{su}(2)$ or su(1,1), has been discussed in detail (Polychronakos 1990, Roček 1991).

The interest of such constructions in the infinite-dimensional case was already noted some years ago in the context of quantum field theory and statistical physics models, where they are known as W-algebras (Zamolodchikov 1986, Schoutens *et al* 1989). More recently, finite versions of these W-algebras were introduced by considering symplectic reductions of finite-dimensional simple Lie algebras (Tjin 1992, de Boer and Tjin 1993). It was shown in particular that the finite $\mathrm{W}_3^{(2)}$-algebra, known as $\overline{\mathrm{W}}_3^{(2)}$, is related to the above-mentioned simplest example of deformed algebra g_d.

The latter has also made recently its appearance in various physical problems. Let us mention three of them. First, the deformed su(2) (or su(1,1)) algebra may be considered as a dynamical symmetry algebra in some quantum many-body models with symmetry-preserving Hamiltonians, such as those occurring in quantum optics (Karassiov 1994, Karassiov and Klimov 1994 and references quoted therein). Next, it is related to generalized deformed parafermions, and through the introduction of a Fermi-like oscillator Hamiltonian, provides a new algebraic description of the bound state spectra of the Morse and Pőschl-Teller potentials (Quesne 1994). Moreover, by superposing these generalized deformed parafermions with ordinary bosons, one gets some deformations of parasupersymmetric quantum mechanics with new and nontrivial properties (Beckers *et al* 1995). Further, the algebra $\overline{\mathrm{W}}_3^{(2)}$ may be considered as the symmetry algebra of the two-dimensional anisotropic harmonic oscillator with frequency ration 2:1 (Bonatsos *et al* 1994).

Motivated by these applications, we shall consider in the present paper another class of examples of deformed algebras g_d, which should be physically relevant. It corresponds to the case where the undeformed subalgebra h is the angular momentum algebra so(3), and the deformed subspace v_d is spanned by the $2\lambda + 1$ components of an so(3) irreducible tensor of rank λ. Special emphasis will be laid on the vector ($\lambda = 1$) and quadrupole ($\lambda = 2$) cases, corresponding to deformations of so(4) and su(3) (or of their noncompact or nonsemisimple variants), respectively.

In the following section, the relevant associativity conditions are established. The vector and quadrupole cases are then studied in detail in sections 3 and 4, respectively. Finally, section 5 contains the conclusion.

2. Nonlinear extensions of so(3)

Let the generators E_i and E_α, introduced in the previous section, be the spherical components $L_m = (-1)^m L^\dagger_{-m}$, $m = +1, 0, -1$, and $T^\lambda_\mu = (-1)^\mu T^{\lambda\dagger}_{-\mu}$, $\mu = \lambda$, $\lambda - 1$, …, $-\lambda$, of an angular momentum operator and of an irreducible tensor of integer rank λ, respectively. In such a case, it is advantageous to write eq. (1.1) in a coupled commutator form as follows:

$$[L, L]^1_m = -\sqrt{2} L_m \tag{2.1a}$$

$$\left[L, T^\lambda\right]^\Lambda_M = -\sqrt{\lambda(\lambda+1)}\, \delta_{\Lambda,\lambda}\, \delta_{M,\mu}\, T^\lambda_\mu \tag{2.1b}$$

$$\left[T^\lambda, T^\lambda\right]^\Lambda_M = f^\Lambda_M(L) \tag{2.1c}$$

where $f^\Lambda(L)$ is an irreducible tensor of rank Λ, whose components can be written as formal power series in the vector operator L. The definition of coupled commutators and some of their properties are reviewed in appendix 1. From eq. (A1.2), it results that the values of Λ in eq. (2.1c) are restricted to odd integers, $\Lambda = 1$, 3, ..., $2\lambda - 1$.

It has been shown by Gaskell *et al* (1978) that the number of linearly independent irreducible tensors of rank Λ, whose components are monomials of degree n in L, is equal to one if $n = \Lambda + 2k$, $k = 0$, 1, 2, ..., and zero otherwise. Hence the explicit form of the functions $f^\Lambda_M(L)$ is given by

$$f^\Lambda_M(L) = \gamma_\Lambda g_\Lambda(\boldsymbol{L}^2)\Big[\cdots\left[[L\times L]^2\times L\right]^3\times\cdots\Big]^\Lambda_M \qquad \Lambda = 1, 3, \ldots, 2\lambda-1 \tag{2.2}$$

where γ_1 is some real normalization constant, $\gamma_\Lambda = 1$ for $\Lambda \neq 1$,

$$g_\Lambda(\boldsymbol{L}^2) = \sum_{k=0}^{\infty} a^{(\Lambda)}_k \boldsymbol{L}^{2k} \qquad a^{(\Lambda)}_k \in \mathbb{R} \qquad a^{(1)}_0 = +1, 0, \text{or} -1 \tag{2.3}$$

is a formal power series in the scalar operator $\boldsymbol{L}^2 = \sum_m (-1)^m L_m L_{-m}$, and the last factor on the right-hand side of (2.2) is a "stretched" product of Λ operators L. The deformed algebra g_d is therefore a $(2K+1)$-th degree algebra, where $K = 0$, 1, 2, ..., provided $a^{(\Lambda)}_k = 0$ if $2k + \Lambda > 2K + 1$, and at least one $a^{(\Lambda)}_k$ with $2k + \Lambda = 2K + 1$ is different from zero. As $K = 0$ corresponds to an ordinary Lie algebra, we shall henceforth refer to K as the deformation order.

We shall be concerned here with the cases where T^λ is an irreducible tensor of rank 1 (vector operator) or rank 2 (quadrupole operator). In the former case, we shall denote T^1_m by A_m, $m = +1$, 0, -1; in eq. (2.1c), Λ then takes the single value $\Lambda = 1$. In the latter case, we shall denote T^2_μ by Q_μ, $\mu = +2$, $+1$, 0, -1, -2; in eq. (2.1c), Λ then runs over the values $\Lambda = 1$ and $\Lambda = 3$. For future use, it is helpful to point out some relations among linearly dependent irreducible tensors:

$$[L\times L]^0_0 = -\tfrac{1}{\sqrt{3}}\boldsymbol{L}^2 \tag{2.4a}$$

$$[L\times L]^1_m = -\tfrac{1}{\sqrt{2}} L_m \tag{2.4b}$$

$$\left[[L\times L]^2\times L\right]^1_m = -\tfrac{2}{\sqrt{15}}\boldsymbol{L}^2 L_m + \tfrac{1}{2}\sqrt{\tfrac{3}{5}} L_m \tag{2.4c}$$

$$\left[[L\times L]^2\times L\right]^2_\mu = -\sqrt{\tfrac{3}{2}}\,[L\times L]^2_\mu. \tag{2.4d}$$

The deformed algebra generated by L_m and T^λ_μ will be associative provided commutators (2.1) satisfy Jacobi identity. For three irreducible tensors T^{λ_1}, U^{λ_2}, V^{λ_3}, of ranks λ_1, λ_2, λ_3, respectively, the latter can be written in a coupled form, as shown in eq. (A1.5). If $\left(T^{\lambda_1}, U^{\lambda_2}, V^{\lambda_3}\right) = (L, L, L)$, $\left(L, L, T^\lambda\right)$, or $\left(L, T^\lambda, T^\lambda\right)$, equation (A1.5) is automatically satisfied, whereas if $\left(T^{\lambda_1}, U^{\lambda_2}, V^{\lambda_3}\right) = \left(T^\lambda, T^\lambda, T^\lambda\right)$, it leads to the set of conditions

$$\sum_{\Lambda_{12}=1,3}^{2\lambda-1} \left\{\delta_{\Lambda_{12},\Lambda_{23}} - 2(-1)^{\lambda-\Lambda} U(\lambda\lambda\Lambda\lambda; \Lambda_{12}\Lambda_{23})\right\} \left[T^\lambda, \left[T^\lambda, T^\lambda\right]^{\Lambda_{12}}\right]^\Lambda_M = 0$$

$$\Lambda_{23} = 1, 3, \ldots, 2\lambda - 1 \qquad \Lambda = |\lambda - \Lambda_{23}|, |\lambda - \Lambda_{23}| + 1, \ldots, \lambda + \Lambda_{23} \quad (2.5)$$

where $U(\lambda\lambda\Lambda\lambda; \Lambda_{12}\Lambda_{23})$ denotes a Racah coefficient in unitary form (Rose 1957). Note that for simplicity's sake, from now on we shall drop the component label of all irreducible tensors.

By using the numerical values of Racah coefficients, one finds that the set of conditions (2.5) reduces to a single independent condition

$$\left[A, [A, A]^1\right]^0 = 0 \qquad (2.6)$$

in the vector case, and to two independent conditions

$$2\sqrt{2}\left[Q, [Q, Q]^1\right]^1 + \sqrt{7}\left[Q, [Q, Q]^3\right]^1 = 0 \qquad (2.7a)$$

$$\left[Q, [Q, Q]^1\right]^3 - 2\left[Q, [Q, Q]^3\right]^3 = 0 \qquad (2.7b)$$

in the quadrupole one. In the next two sections, we shall determine whether these identities are satisfied when the inner commutators are given by eqs. (2.1c), (2.2), and (2.3).

3. The vector case

In the vector case, the deformed algebra g_d is defined by the commutation relations

$$[L, L]^1 = -\sqrt{2}L \qquad [L, A]^\Lambda = -\sqrt{2}\,\delta_{\Lambda,1} A \qquad (3.1a, b)$$

$$[A, A]^1 = f^1(L) = -\sqrt{2}\, g(\boldsymbol{L}^2) L = -\sqrt{2}\left(\sum_{k=0}^\infty a_k \boldsymbol{L}^{2k}\right) L \qquad (3.1c)$$

where the normalization constant γ_1 has been set equal to $-\sqrt{2}$. A K-th order deformation corresponds to $a_K \neq 0$, and $a_{K+1} = a_{K+2} = \cdots = 0$.

3.1. Associativity condition

The algebra g_d is associative provided A satisfies eq. (2.6). By using eqs. (3.1) and (A1.4), the latter can be rewritten as

$$\Big[\big[A, g(\boldsymbol{L}^2)\big]^1 \times L\Big]^0 = 0. \tag{3.2}$$

In the undeformed case where $g(\boldsymbol{L}^2) = a_0$, this condition is trivially fulfilled. Then g_d reduces to an ordinary Lie algebra g. According to whether $a_0 = +1, -1$, or 0, g is the orthogonal algebra so(4), the pseudo-orthogonal algebra so(3,1), or the Euclidian algebra e(3), which is a semidirect sum of an abelian algebra t(3) and so(3) (Biedenharn 1961, Naimark 1964, Bőhm 1979).

For a first-order deformation, condition (3.2) reduces to

$$\Big[\big[A, \boldsymbol{L}^2\big]^1 \times L\Big]^0 = 0. \tag{3.3}$$

From eqs. (2.4a), (A1.4), (3.1b), and (A1.1), one obtains

$$\big[A, \boldsymbol{L}^2\big]^1 = 2\Big\{A + \sqrt{2}\,[L \times A]^1\Big\}. \tag{3.4}$$

Moreover, standard tensor algebra and recoupling techniques (Rose 1957) show that

$$\big[[L \times A]^1 \times L\big]^0 = \big[L \times [L \times A]^1\big]^0 = \big[[L \times L]^1 \times A\big]^0 = -\tfrac{1}{\sqrt{2}}[L \times A]^0 \tag{3.5}$$

where in the last step, use has been made of eq. (2.4b). By introducing eqs. (3.4) and (3.5) into the left-hand side of eq. (3.3), one finds that the latter is identically satisfied.

It is now an easy task to show that if condition (3.2) is fulfilled for a Kth-order deformation, then it is also satisfied for a $(K+1)$th-order one. From eq. (A1.4), one indeed obtains

$$\begin{aligned}\Big[\big[A, \boldsymbol{L}^{2k+2}\big]^1 \times L\Big]^0 &= \Big[\Big[\boldsymbol{L}^{2k} \times \big[A, \boldsymbol{L}^2\big]^1\Big]^1 \times L\Big]^0 + \Big[\Big[\big[A, \boldsymbol{L}^{2k}\big]^1 \times \boldsymbol{L}^2\Big]^1 \times L\Big]^0 \\ &= \boldsymbol{L}^{2k}\Big[\big[A, \boldsymbol{L}^2\big]^1 \times L\Big]^0 + \Big[\big[A, \boldsymbol{L}^{2k}\big]^1 \times L\Big]^0 \boldsymbol{L}^2.\end{aligned} \tag{3.6}$$

Hence, if the relation

$$\left[[A, \boldsymbol{L}^{2k}]^1 \times L\right]^0 = 0 \tag{3.7}$$

is identically satisfied for $k = K$, then the same is true for $k = K + 1$. This completes the proof by induction of the following result:

Proposition 1. For any choice of the deformation parameters a_k, $k = 1, 2, \ldots$, equation (3.1) defines an associative algebra g_d, which is a deformed so(4), so(3,1), or e(3) algebra according to whether $a_0 = +1, -1$, or 0.

Remark. The first-order deformation of so(4) has already been encountered elsewhere. It is indeed the dynamical symmetry algebra of a particle moving in a three-dimensional space with constant curvature under the influence of a Coulomb potential (Higgs 1979, Leemon 1979, Granovskii *et al* 1992, de Vos and van Driel 1993). In such a case, the deformation parameter a_1 is related to the space curvature.

3.2. *Casimir operators*

It is well known that the Lie algebras $g =$ so(4), so(3,1), and e(3) have two independent Casimir operators, which may be written as

$$C_1 = a_0 \boldsymbol{L}^2 + \boldsymbol{A}^2 \qquad C_2 = \boldsymbol{L} \cdot \boldsymbol{A} \tag{3.8}$$

where, as usual, $\boldsymbol{L} \cdot \boldsymbol{A}$ denotes the scalar product $\sum_m (-1)^m L_m A_{-m}$. The purpose of the present subsection is to show that the operators (3.8) can be deformed so as to provide Casimir operators of the deformed algebra g_d.

The case of the second Casimir operator is easily solved. One finds the following result:

Proposition 2. When going from $g =$ so(4), so(3,1), or e(3) to g_d, defined in (3.1), the operator $\boldsymbol{L} \cdot \boldsymbol{A}$ remains a Casimir operator, which we shall denote by C_{2d}.

Proof. By using (A1.4), (3.1), and (A1.2), one obtains

$$[A, \boldsymbol{L} \cdot \boldsymbol{A}]^1 = -\tfrac{1}{\sqrt{2}}[A, A]^1 + \tfrac{1}{\sqrt{2}} f(\boldsymbol{L}^2)[L, L]^1 = 0. \tag{3.9}$$

This completes the proof as, by construction, $\boldsymbol{L} \cdot \boldsymbol{A}$ commutes with L. ∎

The case of the first Casimir operator is more involved and actually has not been solved in full generality. We conjecture that (i) for any choice of the real constants a_k in eq. (3.1), it is possible to find some real constants b_k, $k = 1, 2, \ldots$, such that

$$C_{1d} = h(\boldsymbol{L}^2) + \boldsymbol{A}^2 \qquad \text{where} \qquad h(\boldsymbol{L}^2) = \sum_{k=1}^{\infty} b_k \boldsymbol{L}^{2k} \tag{3.10}$$

is a Casimir operator of the deformed algebra g_d, and (ii) for a Kth-order deformation, these constants are such that $b_k = 0$ for $k > K + 1$. We shall now proceed to prove that this conjecture is at least valid up to fourth order in the deformation.

For such purpose, we have first to determine the commutators of $\boldsymbol{L}^{2k}$ and $\boldsymbol{A}^2$ with A. We state the results in the form of two lemmas.

Lemma 1. For any $k \in \mathbb{N}^+$, the generators of the deformed algebra g_d, defined in (3.1), satisfy the relation

$$\left[A, \boldsymbol{L}^{2k}\right]^1 = \sum_{i=0}^{k-1} x_i^{(k)} \boldsymbol{L}^{2i} A + \sum_{i=0}^{k-1} y_i^{(k)} \boldsymbol{L}^{2i} [L \times A]^1 + \sum_{i=0}^{k-2} z_i^{(k)} \boldsymbol{L}^{2i} \left[[L \times L]^2 \times A\right]^1 \tag{3.11}$$

where $x_i^{(k)}$, $y_i^{(k)}$, $i = 0, 1, \ldots, k-1$, and $z_i^{(k)}$, $i = 0, 1, \ldots, k-2$, are some real constants fulfilling the recursion relations

$$x_i^{(k)} = 2x_i^{(k-1)} + x_{i-1}^{(k-1)} + \tfrac{2}{3}\sqrt{2} y_{i-1}^{(k-1)} + 2\delta_{i,k-1} \qquad i = 0, 1, \ldots, k-1 \tag{3.12a}$$

$$\begin{aligned} y_i^{(k)} = {} & 2\sqrt{2} x_i^{(k-1)} + y_i^{(k-1)} - \sqrt{\tfrac{3}{10}} z_i^{(k-1)} + y_{i-1}^{(k-1)} + 2\sqrt{\tfrac{2}{15}} z_{i-1}^{(k-1)} \\ & + 2\sqrt{2}\delta_{i,k-1} \qquad i = 0, 1, \ldots, k-1 \end{aligned} \tag{3.12b}$$

$$z_i^{(k)} = \sqrt{\tfrac{10}{3}} y_i^{(k-1)} - z_i^{(k-1)} + z_{i-1}^{(k-1)} \qquad i = 0, 1, \ldots, k-2 \tag{3.12c}$$

and the conditions $x_0^{(1)} = 2$, $y_0^{(1)} = 2\sqrt{2}$.

Lemma 2. The generators of the deformed algebra g_d, defined in (3.1), satisfy the relation

$$\left[A, \boldsymbol{A}^2\right]^1 = -\sum_{i=0}^{\infty} u_i \boldsymbol{L}^{2i} A - \sum_{i=0}^{\infty} v_i \boldsymbol{L}^{2i} [L \times A]^1 - \sum_{i=0}^{\infty} w_i \boldsymbol{L}^{2i} \left[[L \times L]^2 \times A\right]^1 \tag{3.13}$$

where u_i, v_i, w_i are some real constants defined by the formal series

$$u_i = \sum_{k=i}^{\infty} a_k \left(2x_i^{(k)} + \tfrac{1}{3}\sqrt{2}y_{i-1}^{(k)} + 2\delta_{k,i}\right) \tag{3.14a}$$

$$v_i = \sum_{k=i}^{\infty} a_k \left(\sqrt{2}x_i^{(k)} + \tfrac{3}{2}y_i^{(k)} - \tfrac{1}{2}\sqrt{\tfrac{3}{10}}z_i^{(k)} + \sqrt{\tfrac{2}{15}}z_{i-1}^{(k)} + 2\sqrt{2}\delta_{k,i}\right) \tag{3.14b}$$

$$w_i = \sum_{k=i+1}^{\infty} a_k \left(\sqrt{\tfrac{5}{6}}y_i^{(k)} + \tfrac{1}{2}z_i^{(k)}\right) \tag{3.14c}$$

in terms of the solution of eq. (3.12).

The proofs of lemmas 1 and 2 are sketched in appendix 2, and the solution obtained for $x_i^{(k)}$, $y_i^{(k)}$, $z_i^{(k)}$, $k \le 5$, by solving eq. (3.12), is listed in table 1.

Finally, by combining lemmas 1 and 2, the following result can be easily derived (for details see appendix 2):

Proposition 3. Up to fourth order in the deformation, the operator C_{1d} defined in eq. (3.10), where

$$\begin{aligned} &b_1 = a_0 + a_1 \qquad b_2 = \tfrac{1}{2}a_1 + \tfrac{4}{3}a_2 - \tfrac{1}{3}a_3 + \tfrac{8}{15}a_4 \qquad b_3 = \tfrac{1}{3}a_3 + \tfrac{5}{3}a_3 - \tfrac{16}{15}a_4 \\ &b_4 = \tfrac{1}{4}a_3 + 2a_4 \qquad b_5 = \tfrac{1}{5}a_4 \qquad b_6 = b_7 = \cdots = 0 \end{aligned} \tag{3.15}$$

is a Casimir operator of the algebra g_d, defined in (3.1).

3.3. Unitary irreducible representations

In the present subsection, we will study the deformations of some unitary irreducible representations (unirreps) of the Lie algebras g = so(4), so(3,1), and e(3) when the latter are replaced by the corresponding deformed algebras g_d.

The unirreps considered are those whose representation space $\mathcal{R}$ contains each of the representation spaces $\mathcal{R}^l$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, of so(3) at most once (Biedenharn 1961, Naimark 1964, Bőhm 1979). Such unirreps can be characterized by

(i) $[p, q]$ where $p \ge |q|$, and p, $|q| \in \mathbb{N}$ or p, $|q| \in \frac{1}{2}\mathbb{N}$, in the so(4) case,

(ii) (l_0, c) where either $l_0 \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$ and $c \in \mathbb{R}$, or $l_0 = 0$ and $c = i\nu$, $\nu \in \mathbb{R}$, in the so(3,1) case,

(iii) (l_0, ϵ) where $l_0 \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$ and $\epsilon \in \mathbb{R}$, in the e(3) case,

where the latter are obtained from those of so(3,1) by an Inönü-Wigner contraction. In cases (i) and (ii) (or (iii)), the decomposition of their representation space is given by

$$\mathcal{R} = \sum_{l=l_0,l_0+1}^{l_1} \oplus \mathcal{R}^l \qquad \text{and} \qquad \mathcal{R} = \sum_{l=l_0,l_0+1}^{\infty} \oplus \mathcal{R}^l \tag{3.16a, b}$$

respectively, where in (3.16a), the minimum (resp. maximum) l value is defined by $l_0 = |q|$ (resp. $l_1 = p$).

The reduced matrix elements of the vector operator A and the eigenvalues of the Casimir operators can be written as†

$$\begin{aligned} \langle [p,q]l \| A \| [p,q]l \rangle &= \frac{q(p+1)}{[l(l+1)]^{1/2}} \\ \langle [p,q]l-1 \| A \| [p,q]l \rangle &= -\left[\frac{(l-q)(l+q)(p+1-l)(p+1+l)}{l(2l-1)}\right]^{1/2} \end{aligned} \tag{3.17}$$

and

$$\langle C_1 \rangle = p(p+2) + q^2 \qquad \langle C_2 \rangle = q(p+1) \tag{3.18}$$

for so(4),

$$\begin{aligned} \langle (l_0,c)l \| A \| (l_0,c)l \rangle &= -\frac{l_0 c}{[l(l+1)]^{1/2}} \\ \langle (l_0,c)l-1 \| A \| (l_0,c)l \rangle &= -\left[\frac{(l-l_0)(l+l_0)(c^2+l^2)}{l(2l-1)}\right]^{1/2} \end{aligned} \tag{3.19}$$

and

$$\langle C_1 \rangle = c^2 - l_0^2 + 1 \qquad \langle C_2 \rangle = -l_0 c \tag{3.20}$$

for so(3,1),

$$\begin{aligned} \langle (l_0,\epsilon)l \| A \| (l_0,\epsilon)l \rangle &= -\frac{l_0 \epsilon}{[l(l+1)]^{1/2}} \\ \langle (l_0,\epsilon)l-1 \| A \| (l_0,\epsilon)l \rangle &= -|\epsilon| \left[\frac{(l-l_0)(l+l_0)}{l(2l-1)}\right]^{1/2} \end{aligned} \tag{3.21}$$

and

$$\langle C_1 \rangle = \epsilon^2 \qquad \langle C_2 \rangle = -l_0 \epsilon \tag{3.22}$$

† The phase convention adopted in the present paper is that of Biedenharn (1961), which differs from that of Naimark (1964) and Bőhm (1979).

for e(3). In all cases, the remaining reduced matrix elements of A can be obtained from the relation

$$\langle l+1\|A\|l\rangle = -\left(\frac{2l+1}{2l+3}\right)^{1/2} \langle l\|A\|l+1\rangle \tag{3.23}$$

valid for any vector operator.

By introducing the function $G(l^2, l_0^2)$, defined by

$$\begin{aligned} G(l^2, l_0^2) &= \frac{1}{l^2 - l_0^2} \sum_{j=l_0}^{l-1} (2j+1) g\big(j(j+1)\big) \\ &= a_0 + \tfrac{1}{2} a_1 (l^2 + l_0^2 - 1) + \tfrac{1}{3} a_2 \left[(l^2-1)^2 + l_0^2(l^2-1) + l_0^2(l_0^2-1)\right] \\ &\quad + \cdots \end{aligned} \tag{3.24}$$

for $l > l_0$, we easily find the following results:

Proposition 4. Provided the deformation parameters a_k, $k > 0$, are chosen in such a way that all quantities under square roots remain nonnegative, the unirreps $[p, q]$, (l_0, c), and (l_0, ϵ) of so(4), so(3,1), and e(3), respectively, can be deformed into unirreps of the corresponding deformed algebras g_d. The reduced matrix elements of A and the eigenvalues of the Casimir operators become

$$\begin{aligned} \langle [p,q]l\|A\|[p,q]l\rangle &= q(p+1) \left[\frac{G\left((p+1)^2, q^2\right)}{l(l+1)}\right]^{1/2} \\ \langle [p,q]l-1\|A\|[p,q]l\rangle &= -\left\{\frac{(l-q)(l+q)\left[(p+1)^2 G\left((p+1)^2, q^2\right) - l^2 G\left(l^2, q^2\right)\right]}{l(2l-1)}\right\}^{1/2} \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \langle C_{1d}\rangle &= h\big(|q|(|q|+1)\big) - (|q|+1) G\left((|q|+1)^2, q^2\right) + (p+1)^2 G\left((p+1)^2, q^2\right) \\ &= \langle C_1\rangle + \tfrac{1}{2} a_1 \Big[\langle C_1\rangle \left(\langle C_1\rangle + 1\right) - \langle C_2\rangle^2\Big] + \tfrac{1}{3} a_2 \langle C_1\rangle \Big[\langle C_1\rangle \left(\langle C_1\rangle + 1\right) - 2\langle C_2\rangle^2\Big] \\ &\quad + \cdots \\ \langle C_{2d}\rangle &= q(p+1) \Big[G\big((p+1)^2, q^2\big)\Big]^{1/2} \\ &= \langle C_2\rangle \Big[1 + \tfrac{1}{2} a_1 \langle C_1\rangle + \tfrac{1}{3} a_2 \left(\langle C_1\rangle^2 - \langle C_2\rangle^2\right) + \cdots\Big]^{1/2} \end{aligned} \tag{3.26}$$

in the deformed so(4) case,

$$\begin{aligned} \langle (l_0,c)l\|A\|(l_0,c)l\rangle &= -\frac{l_0 c}{[l(l+1)]^{1/2}} \\ \langle (l_0,c)l-1\|A\|(l_0,c)l\rangle &= -\left\{\frac{(l-l_0)(l+l_0)\left[c^2 - l^2 G(l^2, l_0^2)\right]}{l(2l-1)}\right\}^{1/2} \end{aligned} \tag{3.27}$$

and

$$
\begin{aligned}
\langle C_{1d} \rangle &= h\big(l_0(l_0+1)\big) - (l_0+1)G\left((l_0+1)^2, l_0^2\right) + c^2 \\
&= \langle C_1 \rangle + \tfrac{1}{2} a_1 l_0^2 \left(l_0^2 - 1\right) + \tfrac{1}{3} a_2 l_0^2 \left(l_0^2 - 1\right)^2 + \cdots \\
\langle C_{2d} \rangle &= \langle C_2 \rangle = -l_0 c
\end{aligned} \tag{3.28}
$$

in the deformed so(3,1) case, while for deformed e(3) they can be obtained by substituting ϵ for c in eqs. (3.27) and (3.28).

Proof. Whenever all a_k's, for which $k > 0$, go to zero, equations (3.25) to (3.28), and their counterparts for e(3) go over into the undeformed results contained in eqs. (3.17) to (3.22), respectively. On the other hand, for arbitrary values of the a_k's satisfying the hypothesis, the validity of the equations can be checked by direct substitution into the commutation relation (3.1c), and the definitions of C_{1d} and C_{2d}. ∎

Remark. For some choices of the deformation parameters, it may happen that the unirreps of g_d considered in proposition 4 do not exhaust the class of unirreps whose representation space contains each of the representation spaces of so(3) at most once. The existence of "extra" representations, which have no counterpart for the undeformed algebra, has already been noted in the deformed su(2) case (Roček 1991).

4. The quadrupole case

In the quadrupole case, the deformed algebra g_d is defined by the commutation relations

$$
[L, L]^1 = -\sqrt{2} L \qquad [L, Q]^\Lambda = -\sqrt{6}\, \delta_{\Lambda,2} Q \tag{4.1a,b}
$$

$$
[Q, Q]^1 = f^1(L) = 3\sqrt{10}\, g_1(\boldsymbol{L}^2) L = 3\sqrt{10} \left(\sum_{k=0}^{\infty} a_k^{(1)} \boldsymbol{L}^{2k} \right) L \tag{4.1c}
$$

$$
\begin{aligned}
[Q, Q]^3 &= f^3(L) = g_3(\boldsymbol{L}^2) \left[[L \times L]^2 \times L \right]^3 \\
&= \left(\sum_{k=0}^{\infty} a_k^{(3)} \boldsymbol{L}^{2k} \right) \left[[L \times L]^2 \times L \right]^3
\end{aligned} \tag{4.1d}
$$

where the normalization constant γ_1 has been set equal to $3\sqrt{10}$. The algebra is associative provided Q satisfies eqs. (2.7a,b).

In the undeformed case, where the algebra g_d reduces to an ordinary Lie algebra g, one has $a_k^{(1)} = 0$, $k = 1, 2, \ldots$, and $a_k^{(3)} = 0$, $k = 0, 1, \ldots$. According to whether $a_0^{(1)} = +1, -1$, or 0, g is the special unitary algebra su(3) (Elliott 1958a,b), the special linear algebra sl(3,$\mathbb{R}$) (Weaver and Biedenharn 1972), or the semidirect sum of an abelian algebra t(5) and so(3) (Ui 1970, Weaver *et al* 1973).

If we restrict ourselves to a first-order deformation, equations (4.1c) and (4.1d) become

$$[Q,Q]^1 = 3\sqrt{10}\,\epsilon L + \alpha \boldsymbol{L}^2 L \tag{4.2a}$$

$$[Q,Q]^3 = \beta \left[[L \times L]^2 \times L\right]^3 \tag{4.2b}$$

where α, β, and ϵ are defined by $\alpha = 3\sqrt{10} a_1^{(1)}$, $\beta = a_0^{(3)}$, and $\epsilon = a_0^{(1)} = +1, -1$, or 0. By inserting eqs. (4.2a,b) into eqs. (2.7a,b) and taking eq. (4.1b) into account, we obtain the two associativity conditions

$$2\sqrt{2}\,\alpha \left[Q, \boldsymbol{L}^2 L\right]^1 + \sqrt{7}\,\beta \left[Q, \left[[L \times L]^2 \times L\right]^3\right]^1 = 0 \tag{4.3a}$$

$$\alpha \left[Q, \boldsymbol{L}^2 L\right]^3 - 2\beta \left[Q, \left[[L \times L]^2 \times L\right]^3\right]^3 = 0. \tag{4.3b}$$

Straightforward tensor algebra leads to the following results:

$$\begin{aligned}
\left[Q, \boldsymbol{L}^2 L\right]^\Lambda &= 2\sqrt{3}\Big\{\left[\sqrt{3} - U(11\Lambda2;12)\right][L \times Q]^\Lambda \\
&\quad + \sqrt{2}\,U(11\Lambda2;22)\left[[L \times L]^2 \times Q\right]^\Lambda\Big\} \\
\left[Q, \left[[L \times L]^2 \times L\right]^3\right]^\Lambda &= \sqrt{3}\Big\{\big[\sqrt{7}\,U(22\Lambda1;23) \\
&\quad - 2\sqrt{2\Lambda(\Lambda+1)}\,U(22\Lambda1;\Lambda3)U(21\Lambda1;22)\big][L \times Q]^\Lambda \\
&\quad + 3\sqrt{2}\,U(21\Lambda2;23)\left[[L \times L]^2 \times Q\right]^\Lambda\Big\}
\end{aligned} \tag{4.4}$$

valid for $\Lambda = 1$ or 3. By replacing Racah coefficients by their numerical values in eq. (4.4), conditions (4.3a,b) can be rewritten as

$$\left(6\sqrt{10}\,\alpha - 7\beta\right)[L \times Q]^1 + \sqrt{6}\left(2\sqrt{10}\,\alpha + 7\beta\right)\left[[L \times L]^2 \times Q\right]^1 = 0 \tag{4.5a}$$

$$2\left(\sqrt{10}\,\alpha + 3\beta\right)[L \times Q]^3 + \left(\sqrt{10}\,\alpha - 9\beta\right)\left[[L \times L]^2 \times Q\right]^3 = 0. \tag{4.5b}$$

Since they cannot be satisfied for any choice of the deformation parameters α, β, we conclude that for a first-order deformation, the algebra g_d, defined in (4.1), is not associative contrary to what happens for the algebra (3.1) in the vector case. We shall therefore pursue the analysis of the quadrupole case no further.

5. Conclusion

In the present paper, we established that there exist deformations of the Lie algebras so(4), so(3,1), and e(3) that leave their so(3) subalgebra undeformed and preserve their coset structure. We proved that the Casimir operators of these Lie algebras can be deformed so as to provide corresponding operators for the deformed algebras. Moreover, we constructed those unirreps of the latter that go over for vanishing deformation into the unirreps of the former belonging to an important class of representations.

In contrast, we showed that a similar deformation of the Lie algebras su(3), sl(3,$\mathbb{R}$), and of the semidirect sum of t(5) and so(3) is not possible because the associativity conditions are violated in first order in the deformation.

It should be stressed that the deformations of so(4), so(3,1), and e(3) studied here differ from the standard q-algebras $\mathrm{so}_q(4)$, $\mathrm{so}_q(3,1)$, $\mathrm{e}_q(3)$ (Drinfeld 1986, Jimbo 1985, Celeghini *et al* 1991, Chakrabarti 1993), as well as from an alternative deformation of the orthogonal and pseudo-orthogonal Lie algebras proposed by Gavrilik and Klimyk (1991) (see also Gavrilik 1993). In both these approaches, the so(3) subalgebra is indeed deformed contrary to what happens in the present one. Since rotational invariance and angular momentum conservation are important properties of many physical systems, one may hope that the deformed algebras introduced in this paper will prove more relevant to applications than those previously considered.

Some problems wherein the coset structure so(4)/so(3) is important can be found in standard quantum mechanics (e.g., that of a particle in a Coulomb potential (Biedenharn 1961)), as well as in parasupersymmetric quantum mechanics with three parasupercharges (Debergh and Nikitin 1995). Deformations of so(4) preserving that coset structure may therefore be expected to play a role in similar contexts. It is already known that the first-order deformation of so(4) is the symmetry algebra of a particle in a Coulomb potential when the space has a constant curvature (Higgs 1979, Leemon 1979, Granovskii *et al* 1992, de Vos and van Driel 1993). All the general results derived in the present paper therefore apply to such a problem. At a more phenomenological level, deviations from hydrogenic spectra that are found for many-electron atoms or excitons in semiconductors might be accounted for by some deformations of so(4). Similarly, deformations of parasupersymmetric quantum

mechanics with three parasupercharges might lead to some parasupersymmetric Hamiltonians with new and nontrivial properties, as happens in the case of two parasupercharges (Debergh *et al* 1995). We hope to come back to some of these problems in forthcoming publications.

Appendix 1. Definition and properties of coupled commutators

The purpose of this appendix is to review the definition and some useful properties of coupled commutators.

The coupled commutator of two so(3) irreducible tensors T^{λ_1} and U^{λ_2}, of ranks λ_1 and λ_2 respectively, is defined by

$$\left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda}_{M} = \sum_{\mu_1\mu_2} \langle\lambda_1\mu_1, \lambda_2\mu_2|\Lambda M\rangle \left[T^{\lambda_1}_{\mu_1}, U^{\lambda_2}_{\mu_2}\right] \tag{A1.1}$$

in terms of an ordinary commutator $[\,,\,]$ and of an su(2) Wigner coefficient $\langle\,,\,|\,\rangle$. By using a symmetry property of the latter (Rose 1957), equation (A1.1) can be alternatively written as

$$\begin{aligned}\left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda}_{M} &= \left[T^{\lambda_1} \times U^{\lambda_2}\right]^{\Lambda}_{M} - (-1)^{\lambda_1+\lambda_2-\Lambda} \left[U^{\lambda_2} \times T^{\lambda_1}\right]^{\Lambda}_{M} \\ &= -(-1)^{\lambda_1+\lambda_2-\Lambda} \left[U^{\lambda_2}, T^{\lambda_1}\right]^{\Lambda}_{M}\end{aligned} \tag{A1.2}$$

where

$$\left[T^{\lambda_1} \times U^{\lambda_2}\right]^{\Lambda}_{M} = \sum_{\mu_1\mu_2} \langle\lambda_1\mu_1, \lambda_2\mu_2|\Lambda M\rangle\, T^{\lambda_1}_{\mu_1} U^{\lambda_2}_{\mu_2}. \tag{A1.3}$$

For three irreducible tensors T^{λ_1}, U^{λ_2}, V^{λ_3}, the well-known relation $[A, BC] = [A, B]C + B[A, C]$ becomes in coupled form

$$\begin{aligned}\left[T^{\lambda_1}, \left[U^{\lambda_2} \times V^{\lambda_3}\right]^{\Lambda_{23}}\right]^{\Lambda}_{M} &= \sum_{\Lambda_{12}} U(\lambda_1\lambda_2\Lambda\lambda_3; \Lambda_{12}\Lambda_{23}) \left[\left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda_{12}} \times V^{\lambda_3}\right]^{\Lambda}_{M} \\ &\quad + \sum_{\Lambda_{13}} (-1)^{\lambda_3+\Lambda-\Lambda_{13}-\Lambda_{23}} U(\lambda_1\lambda_3\Lambda\lambda_2; \Lambda_{13}\Lambda_{23}) \\ &\quad \times \left[U^{\lambda_2} \times \left[T^{\lambda_1}, V^{\lambda_3}\right]^{\Lambda_{13}}\right]^{\Lambda}_{M}\end{aligned} \tag{A1.4}$$

where $U(\lambda_1\lambda_2\Lambda\lambda_3; \Lambda_{12}\Lambda_{23})$ denotes a Racah coefficient in unitary form (Rose 1957). In the same way, the Jacobi identity $\left[A, [B, C]\right] + \left[B, [C, A]\right] + \left[C, [A, B]\right] = 0$ can be rewritten as

$$\begin{aligned}&\left[T^{\lambda_1}, \left[U^{\lambda_2}, V^{\lambda_3}\right]^{\Lambda_{23}}\right]^{\Lambda}_{M} + \sum_{\Lambda_{31}} (-1)^{\lambda_1+\Lambda_{23}-\Lambda} U(\lambda_2\lambda_3\Lambda\lambda_1; \Lambda_{23}\Lambda_{31}) \left[U^{\lambda_2}, \left[V^{\lambda_3}, T^{\lambda_1}\right]^{\Lambda_{31}}\right]^{\Lambda}_{M} \\ &\qquad + \sum_{\Lambda_{12}} (-1)^{\lambda_3+\Lambda_{12}-\Lambda} U(\lambda_1\lambda_2\Lambda\lambda_3; \Lambda_{12}\Lambda_{23}) \left[V^{\lambda_3}, \left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda_{12}}\right]^{\Lambda}_{M} \\ &= 0.\end{aligned} \tag{A1.5}$$

Appendix 2. Proofs of lemmas 1 and 2 and of proposition 3

To prove eqs. (3.11) and (3.12), we proceed by induction over k. For the lowest value $k=1$, equation (3.11) reduces to (3.4). For $k>1$, we start from the identities

$$\left[A, \boldsymbol{L}^{2k}\right]^1 = \boldsymbol{L}^{2k-2}\left[A, \boldsymbol{L}^2\right]^1 + \boldsymbol{L}^2\left[A, \boldsymbol{L}^{2k-2}\right]^1 + \left[\left[A, \boldsymbol{L}^{2k-2}\right]^1, \boldsymbol{L}^2\right]^1 \tag{A2.1}$$

and

$$\left[\left[A, \boldsymbol{L}^{2k-2}\right]^1, \boldsymbol{L}^2\right]^1 = 2\left\{\left[A, \boldsymbol{L}^{2k-2}\right]^1 + \sqrt{2}\left[L \times \left[A, \boldsymbol{L}^{2k-2}\right]^1\right]^1\right\} \tag{A2.2}$$

resulting from eq. (A1.4) and standard tensor algebra. Then assuming eq. (3.11) to be valid when k is replaced by $k-1$, we use it to compute the right-hand sides of eqs. (A2.1) and (A2.2). The result contains two tensor products that are not in standard form, as those appearing on the right-hand side of (3.11), but can be rewritten in such a form by using the following identities:

$$\left[L \times [L \times A]^1\right]^1 = \tfrac{1}{3}\boldsymbol{L}^2 A - \tfrac{1}{2\sqrt{2}}[L \times A]^1 + \tfrac{1}{2}\sqrt{\tfrac{5}{3}}\left[[L \times L]^2 \times A\right]^1 \tag{A2.3}$$

$$\left[L \times \left[[L \times L]^2 \times A\right]^1\right]^1 = -\tfrac{1}{4\sqrt{15}}\left(3 - 4\boldsymbol{L}^2\right)[L \times A]^1 - \tfrac{3}{2\sqrt{2}}\left[[L \times L]^2 \times A\right]^1. \tag{A2.4}$$

After some straightforward calculations, equation (3.11) is obtained provided $x_i^{(k)}$, $y_i^{(k)}$, and $z_i^{(k)}$ satisfy eq. (3.12).

The proof of eqs. (3.13) and (3.14) is based upon the identities

$$\left[A, \boldsymbol{A}^2\right] = -2\sqrt{2}\sum_{k=0}^{\infty} a_k \boldsymbol{L}^{2k}[L \times A]^1 + \sqrt{2}\sum_{k=0}^{\infty} a_k\left[A, \boldsymbol{L}^{2k}L\right]^1 \tag{A2.5}$$

and

$$\left[A, \boldsymbol{L}^{2k}L\right]^1 = -\sqrt{2}\left[A, \boldsymbol{L}^{2k}\right]^1 - \left[L \times \left[A, \boldsymbol{L}^{2k}\right]^1\right]^1 - \sqrt{2}\boldsymbol{L}^{2k}A \tag{A2.6}$$

resulting from eqs. (A1.4) and (3.1), as well as standard tensor algebra. Taking eqs. (3.11), (A2.3), and (A2.4) into account directly leads to the searched for results.

Finally, from eqs. (3.10), (3.11), and (3.13), it follows that for arbitrary constants a_k, the condition $[A, C_{1d}]^1 = 0$ is fulfilled provided

$$\sum_{k=i+1}^{\infty} b_k x_i^{(k)} - u_i = \sum_{k=i+1}^{\infty} b_k y_i^{(k)} - v_i = \sum_{k=i+2}^{\infty} b_k z_i^{(k)} - w_i = 0 \qquad i = 0, 1, 2, \ldots. \tag{A2.7}$$

By successively using eqs. (3.14) and (3.12), these conditions can be rewritten as

$$2\sum_{k=i+1}^{\infty} b'_k x_i^{(k)} = \sum_{k=i}^{\infty} a_k \left(2x_i^{(k)} - x_{i-1}^{(k)} + 2\delta_{k,i}\right) \qquad i=0,1,2,\ldots \tag{A2.8a}$$

$$2\sum_{k=i+1}^{\infty} b'_k y_i^{(k)} = \sum_{k=i}^{\infty} a_k \left(2y_i^{(k)} - y_{i-1}^{(k)} + 2\sqrt{2}\delta_{k,i}\right) \qquad i=0,1,2,\ldots \tag{A2.8b}$$

$$2\sum_{k=i+2}^{\infty} b'_k z_i^{(k)} = \sum_{k=i+1}^{\infty} a_k \left(2z_i^{(k)} - z_{i-1}^{(k)}\right) \qquad i=0,1,2,\ldots \tag{A2.8c}$$

where $b'_k \equiv b_k - \frac{1}{2}a_{k-1}$.

If we restrict ourselves to a Kth-order deformation and assume that $b_k = 0$ for $k > K+1$, conditions (A2.8a) and (A2.8b) reduce to two systems of $K+1$ equations (corresponding to $i = 0,\ 1,\ \ldots,\ K$) in $K+1$ unknowns b'_k, $k = 1,\ 2,\ \ldots,\ K+1$, while condition (A2.8c) leads to a system of K equations (corresponding to $i = 0$, 1, …, $K-1$) in K unknowns b'_k, $k = 2,\ 3,\ \ldots,\ K+1$. It only remains to solve (A2.8a) and to check that its solution also satisfies the two remaining systems of equations. This calculation was carried out for $K = 4$, and the results are contained in eq. (3.15).

References

Beckers J, Debergh N and Quesne C 1995 Parasupersymmetric quantum mechanics with generalized deformed parafermions *Preprint* ULB-ULG

Biedenharn L C 1961 *J. Math. Phys.* **2** 433

Bőhm A 1979 *Quantum Mechanics* (Berlin: Springer) p 146

Bonatsos D, Daskaloyannis C, Kolokotronis P and Lenis D 1994 Nonlinear extension of the u(2) algebra as the symmetry algebra of the planar anisotropic quantum harmonic oscillator with rational ratio of frequencies and "pancake" nuclei *Preprint* NUCL-TH/9412003

Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1159

Chakrabarti A 1993 *J. Math. Phys.* **34** 1964

Debergh N and Nikitin A G 1995 Parasupersymmetric quantum mechanics with an arbitrary number of parasupercharges and orthogonal Lie algebras *Helv. Phys. Acta* in press

de Boer J and Tjin T 1993 *Commun. Math. Phys.* **158** 485

de Vos K and van Driel P 1993 The Kazhdan-Lusztig conjecture for finite W-algebras *Preprint* hep-th/9312016

Drinfeld V G 1986 *Proc. Int. Congr. of Mathematicians (Berkeley, CA)* ed. A M Gleason (Providence, RI: AMS) p 798

Elliott J P 1958a *Proc. R. Soc.* A **245** 128

——1958b *Proc. R. Soc.* A **245** 562

Fairlie D B and Nuyts J 1994 *J. Math. Phys.* **35** 3794

Fairlie D B and Zachos C K 1991 *Phys. Lett.* **256B** 43

Gaskell R, Peccia A and Sharp R T 1978 *J. Math. Phys.* **19** 727

Gavrilik A M 1993 *Theor. Math. Phys.* **95** 546

Gavrilik A M and Klimyk A U 1991 *Lett. Math. Phys.* **21** 215

Granovskii Ya I, Zhedanov A S and Lutsenko I M 1992 *Theor. Math. Phys.* **91** 604

Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309

Jimbo M 1985 *Lett. Math. Phys.* **10** 63

Karassiov V P 1994 *J. Phys. A: Math. Gen.* **27** 153

Karassiov V P and Klimov A B 1994 *Phys. Lett.* **191A** 117

Leemon H I 1979 *J. Phys. A: Math. Gen.* **12** 489

Naimark M A 1964 *Linear Representations of the Lorentz Group* (New York: Pergamon)

Polychronakos A P 1990 *Mod. Phys. Lett.* A **5** 2325

Quesne C 1994 *Phys. Lett.* **193A** 245

Roček M 1991 *Phys. Lett.* **255B** 554

Rose M E 1957 *Elementary Theory of Angular Momentum* (New York: Wiley)

Schoutens K, Sevrin A and van Nieuwenhuizen P 1989 *Commun. Math. Phys.* **124** 87

Tjin T 1992 *Phys. Lett.* **292B** 60

Ui H 1970 *Prog. Theor. Phys.* **44** 153

Weaver L and Biedenharn L C 1972 *Nucl. Phys.* A **185** 1

Weaver L, Biedenharn L C and Cusson R Y 1973 *Ann. Phys., NY* **77** 250

Zamolodchikov A B 1986 *Theor. Math. Phys.* **65** 1205

Tables and table captions

Table 1. Solution of the recursion relations (3.12) up to $k = 5$.

	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
$x_i^{(2)}$	4	$\frac{20}{3}$	—	—	—
$y_i^{(2)}$	$6\sqrt{2}$	$4\sqrt{2}$	—	—	—
$z_i^{(2)}$	$4\sqrt{\frac{5}{3}}$	—	—	—	—
$x_i^{(3)}$	8	$\frac{76}{3}$	14	—	—
$y_i^{(3)}$	$12\sqrt{2}$	$26\sqrt{2}$	$6\sqrt{2}$	—	—
$z_i^{(3)}$	$8\sqrt{\frac{5}{3}}$	$4\sqrt{15}$	—	—	—
$x_i^{(4)}$	16	$\frac{224}{3}$	88	24	—
$y_i^{(4)}$	$24\sqrt{2}$	$88\sqrt{2}$	$68\sqrt{2}$	$8\sqrt{2}$	—
$z_i^{(4)}$	$16\sqrt{\frac{5}{3}}$	$16\sqrt{15}$	$8\sqrt{15}$	—	—
$x_i^{(5)}$	32	$\frac{592}{3}$	368	$\frac{680}{3}$	$\frac{110}{3}$
$y_i^{(5)}$	$48\sqrt{2}$	$248\sqrt{2}$	$352\sqrt{2}$	$140\sqrt{2}$	$10\sqrt{2}$
$z_i^{(5)}$	$32\sqrt{\frac{5}{3}}$	$48\sqrt{15}$	$160\sqrt{\frac{5}{3}}$	$40\sqrt{\frac{5}{3}}$	—